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Identification of invariant measures of interacting systems

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Abstract

In this paper we provide an approach for identifying certain mixture representations of some invariant measures of interacting stochastic systems. This is related to the problem of ergodicity of certain extremal invariant measures that are translation invariant. Corresponding to these, results concerning the existence of invariant measures and certain weak convergence of the systems are also provided.

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1. Introduction

The problem considered in this paper is motivated by investigating an open question for interacting particle systems. The question is roughly that: for a certain system with configuration space $E = W^{Z^d}$, if an invariant probability measure ν is translation invariant and extremal among all such measures, is it ergodic? See [1, 7] for a precise description. Supported by a well-known result for Gibbs measures, the answer to the above question is expected to be affirmative under very mild conditions. But only partial results were obtained in some special cases (cf [1, 2, 7]). In this paper we provide an approach for studying related problems, more precisely, for identifying certain mixture representations of an invariant measure. This approach is hopefully applicable for more widely ranging problems.

Let W be a Polish space endowed with the Borel σ-algebra, representing the state space of the particle at a single site, S a countable set, serving as the set of sites, $E = W^S$ endowed with the product topology and the corresponding Borel σ-algebra, interpreted as the configuration space. $M_1(E)$ denotes the space of all probability measures on E, endowed with the weak topology. In this paper we will mainly handle the case $S = Z^d$, the d-dimensional lattice, but extension to more general cases only requires more notation. The time evolution of an interacting particle system can usually be described by a homogeneous Feller–Markov process $\{P_\eta, \eta \in E\}$ on $\Omega = D([0,\infty), E)$, driven by certain dynamic characteristics, where Ω is the space of functions from $[0,\infty)$ to E that are right continuous with left limits. For a precise description of such Markov processes in the case when $W = \{0, 1\}$, see [7]. Let $\{S(t), t \geqslant 0\}$ be the corresponding Markov semigroup on $C_b(E)$—the space of bounded continuous functions on E. From the point of view of statistical physics, one of the most important objects associated with such a Markov process is the set of its invariant probability measures, denoted by $M_i(E)$. Recall that $\mu \in M_i(E)$ iff $\mu S(t) = \mu$ for every $t \geqslant 0$, where $\mu S(t)$ is the distribution of the process at time t starting with the initial distribution μ. Phase transition can be described by the non-uniqueness of such invariant measures. If there is a unique invariant measure ν and, starting from any initial distribution μ, $\mu S(t)$ converges in a certain sense to ν as $t \to \infty$, then the system can be said to be ergodic (cf [7]). Thus characterizing invariant measures of a stochastic system is of fundamental interest. Clearly, those extremal invariant measures play an essential role. Denote by $M_{\mathrm{i,e}}(E)$ the set of all such extremal measures.

If the dynamic characteristics driving the system are (space-) translation invariant, then most of the typical invariant measures should also be translation invariant. Thus a natural problem is to characterize those invariant measures that are (space-shift) ergodic. Let $M_s(E)$ be the set of translation invariant probability measures on E and $M_{s,e}(E)$ the set of ergodic measures in $M_s(E)$ and $(M_i(E) \cap M_s(E))_e$ the set of extremal measures in $M_i(E) \cap M_s(E)$. Then the question stated in the first paragraph can be formally stated as: is every $\nu \in (M_i(E) \cap M_s(E))_e$ ergodic? By the well-known ergodic decomposition theorem, for every $\nu \in (M_i(E) \cap M_s(E))_e$, there is a unique probability measure Γ on $M_s(E)$, supported in $M_{s,e}(E)$, such that

$$\nu = \int_{M_{s,e}(E)} \mu \Gamma(\mathrm{d}\mu). \tag{1.1}$$

Obviously, if one can identify Γ-almost all component measures in the above mixture as invariant measures of the process, then the question is affirmatively answered. In this paper we provide an approach for studying related problems from this point of view. Actually we consider identifying both the component and the mixing measures. We summarize as follows the main results and the techniques used in proving them.

In section 2 consider the problem of identifying the mixing measure. This is done by considering the dynamic cases. That is, we consider continuous-time Markov processes, focusing on characterizing their invariant measures. In theorem 2.1, we first present a connection between the tightness of T_t^μ, the average in time of laws of the process, and that of R_t^μ, the laws of the empirical measures of the process. Then we give a representation relating their limits. Furthermore, we show that if T_t^μ converges weakly to some extremal invariant measure ν of the process, then R_t^μ converges weakly to the point mass δ_ν and in particular, if μ is itself an invariant measure, then R_t^μ converges weakly to a measure supported in the set of extremal invariant measures.

The above results suggest a way of identifying the mixing measure in a representation such as (1.1). In theorem 2.2 we carry out this idea by using a sample of n iid paths that run up to time n. We show that a certain average Q_n^* of their empirical measures converges to such a mixing measure in probability. As a consequence we see in corollary 2.3 that if T_t^μ converges to an extremal invariant measure ν, then Q_n^* is a consistent estimator of δ_ν and in particular, if μ is itself an invariant measure, then Q_n^* is a consistent estimator of the mixing measure in (1.1). Proposition 2.4 provides a useful sufficient condition for the required tightness.

Section 3 is devoted to the problem of identifying the component measures in a mixture. This is carried out from the point of view of a random field, focusing on the space of probability measures having certain invariance. We use the asymptotic behaviour of the empirical fields R_n to study the properties of the component measures. The desired property is characterized by a certain functional which governs the large deviation upper bounds of the laws of the empirical fields.

Applications are given in section 3 and, especially, in section 4. We show in section 3 that under some mild conditions, every stationary Markov chain on a Polish space is a mixture of those stationary and ergodic chains with the same transition probabilities. In section 4 we mainly discuss applications to interacting particle systems, or more precisely, to Gibbs random fields and related stochastic Ising models. We first provide, using our general results in sections 2 and 3, a short proof of a well-known result which says that every translation invariant Gibbs measure is a mixture of those Gibbs measures that are ergodic. We then describe a way of approaching the extremal Gibbs measures relative to a certain potential from the point of view of Glauber dynamics. We formulate certain consistent estimators of such measures using some independent paths of the dynamic process. This suggests a way of inferring the existence of phase transitions.

Our approach involves certain tightness and large deviation (LD) arguments. Such arguments turn out to be useful in proving the existence of invariant measures and certain weak convergence of the system (see section 2). We use large deviation estimates both to specify the support of the mixing measure and to characterize properties of the component measures.

Throughout this paper, for a measure μ and a function f, both $E^\mu f$ and $\mu(f)$ denote the expectation of f w.r.t. μ.

2. Identifying the mixing measure

In this section we consider the problem of identifying the mixing measure by studying the dynamic case. We will not specify the dynamic characteristics of the systems under consideration. We only assume that they determine a unique homogeneous Feller–Markov process. Let $\{P_\eta, \eta \in E\}$ be the Markov process on Ω that we are interested in. For $\mu \in M_1(E)$, define

$$P_\mu = \int P_\eta \mu(\mathrm{d}\eta)$$

which is the Markov process with initial distribution μ. Then $\mu S(t)$ is just $P_\mu(\eta_t \in \cdot)$. Define for $t > 0$

$$T_t^\mu = \frac{1}{t}\int_0^t \mu S(u)\,\mathrm{d}u.$$

The empirical measure L_t^μ of the process up to time t is defined as

$$L_t = L_t(\eta_\cdot) = \frac{1}{t}\int_0^t \delta_{\eta_u}\,\mathrm{d}u \qquad \eta_\cdot \in \Omega$$

where δ_η is the Dirac measure centred at $\eta \in E$. Let $Q_t^\mu = P_\mu(L_t \in \cdot)$ be the distribution of L_t under P_μ, which is an element in $M_1(M_1(E))$, the set of probability measures on $M_1(E)$. Now we can state our first main result.

Theorem 2.1. *Given $\mu \in M_1(E)$.*

(1) If $\{Q_t^\mu, t > 0\}$ is tight, then so is $\{T_t^\mu, t > 0\}$. Thus $M_i(E) \neq \emptyset$. Furthermore, every weak limit Q_∞^μ of Q_t^μ as $t \to \infty$ is supported in $M_i(E)$ and every weak limit μ_∞ of T_t^μ as $t \to \infty$ admits a mixture representation as

$$\mu_\infty = \int_{M_i(E)} \gamma\, Q_\infty^\mu(\mathrm{d}\gamma)$$

where Q_∞^μ is a weak limit of Q_t^μ as $t \to \infty$.

(2) If $\{Q_t^\mu, t > 0\}$ *is tight and* $\lim_{t\to\infty} T_t^\mu = \nu$ *weakly for some* $\nu \in M_{\mathrm{i,e}}(E)$*, then*

$$\lim_{t\to\infty} Q_t^\mu = \delta_\nu$$

weakly. Furthermore, if $\mu \in M_i(E)$*, then there exists the weak limit*

$$Q_\infty^\mu = \lim_{t\to\infty} Q_t^\mu$$

which is supported in $M_{\mathrm{i,e}}(E)$ *and*

$$\mu = \int_{M_{\mathrm{i,e}}(E)} \gamma\, Q_\infty^\mu(\mathrm{d}\gamma). \tag{2.1}$$

Proof. To prove conclusion (1), fix an open set $U \supset M_i(E)$. For any $\epsilon > 0$, choose a compact subset K_ϵ of $M_1(E)$ such that $Q_t^\mu(K_\epsilon^c) < \epsilon, \forall t > 0$. To estimate $Q_t^\mu(U^c \cap K_\epsilon)$, we note that from a well-known large deviation result, there exists a lower semi-continuous function I from $M_1(E)$ to $[0, \infty]$, such that $I(\gamma) = 0$ iff $\gamma \in M_i(E)$ and that

$$\limsup_{t\to\infty} \frac{1}{t} \log Q_t^\mu(U^c \cap K_\epsilon) \leqslant -\inf_{\gamma\in U^c\cap K_\epsilon} I(\gamma) < 0.$$

Combining the above estimates we see that

$$\limsup_{t\to\infty} Q_t^\mu(U^c) \leqslant \epsilon$$

for every $\epsilon > 0$. This implies that every weak limit of Q_t^μ as $t \to \infty$ is supported in $M_i(E)$. Now note that for every $f \in C_b(E)$,

$$T_t^\mu(f) = \int L_t(f)\, \mathrm{d}P_\mu = \int \alpha(f)\, \mathrm{d}Q_t^\mu.$$

The tightness of T_t^μ follows from this and the tightness of Q_t^μ. Then a standard argument gives (2.1).

Now we prove conclusion (2). First suppose that Q_t^μ converges weakly to some $\nu \in M_{\mathrm{i,e}}(E)$ as $t \to \infty$. By the assumptions, the representation

$$T_{t^\mu} = \int \alpha\, Q_t^\mu(\mathrm{d}\alpha)$$

and conclusion (1) we see that any weak limit Γ of $\{Q_t^\mu, t \geqslant 0\}$ satisfies

$$\nu = \int \alpha \Gamma(\mathrm{d}\alpha).$$

It follows from the extremality of ν that $\Gamma = \delta_\nu$. This implies that the weak limit $\lim_{t\to\infty} Q_t^\mu$ exists and is just δ_ν.

Now suppose $\mu \in M_i(E)$. Then from [2] we know that μ can be uniquely represented as

$$\mu = \int_{\gamma\in M_{\mathrm{i,e}}(E)} \gamma \Gamma(\mathrm{d}\gamma) \tag{2.2}$$

for some probability measure Γ on $M_i(E)$ supported in $M_{\mathrm{i,e}}(E)$. It then follows that

$$T_t^\mu = \int_{\gamma\in M_{\mathrm{i,e}}(E)} T_t^\gamma \Gamma(\mathrm{d}\gamma)$$

and $T_t^\gamma = \gamma$. Thus from conclusion (1) we know that $\lim_{t\to\infty} Q_t^\gamma = \delta_\gamma$ weakly for Γ-almost every γ. It follows that

$$\lim_{t\to\infty} Q_t^\mu = \lim_{t\to\infty} \int Q_t^\gamma \Gamma(\mathrm{d}\gamma) = \int \delta_\gamma \Gamma(\mathrm{d}\gamma).$$

The proof is completed by setting $Q_\infty^\mu = \int \delta_\gamma \Gamma(\mathrm{d}\gamma)$. $\square$

To provide an application of theorem 2.1 to the problem of identifying the mixing measure Q_∞^μ in (2.1), we need the following theorem. The idea is to estimate the 'parameter' from some realization of the paths of the process. Let $\{\eta_t, 0 \leqslant t \leqslant n\}$ be such a path up to time n, $\{\eta_t^{(i)}, 0 \leqslant t \leqslant n\}$, $i = 1, \dots, n$, be n independent copies of the above path. $P_\mu^{(n)}$ denote the n-fold product of P_μ. Define

$$L_n^i = \frac{1}{n}\int_0^n \delta_{\eta_t^{(i)}}\,\mathrm{d}t \qquad i = 1, \dots, n$$

and

$$Q_n^* = \frac{1}{n}\sum_{i=1}^n \delta_{L_n^i}$$

which is a probability measure on $M_1(E)$. Then we have the following

Theorem 2.2. *Given* $\mu \in M_i(E)$. *If* $\{P_\mu^{(n)}(Q_n^* \in \cdot), n \geqslant 1\}$ *is tight and* $\lim_{t\to\infty} Q_t^\mu = Q_\infty^\mu$ *weakly, then* $\lim_{n\to\infty} Q_n^* = Q_\infty^\mu$ *weakly in probability* $P_\mu^{(n)}$.

Remark 2.1. This theorem implies that the Q_n^* are consistent estimators of Q_∞^μ.

Proof of theorem 2.2. Let $d(\cdot\,,\cdot)$ be any fixed metric on $M_1(M_1(E))$ that generates the weak topology. For any $\epsilon > 0$ and $\delta > 0$, choose a compact subset K_ϵ in $M_1(M_1(E))$ such that $P_\mu^{(n)}(Q_n^* \in K_\epsilon^c) < \epsilon\, \forall n \geqslant 1$. Thus

$$P_\mu^{(n)}\big(d\big(Q_n^*, Q_\infty^\mu\big) \geqslant \delta\big) < P_\mu^{(n)}(d(Q_n^*, Q_\infty^*) \geqslant \delta;\ Q_n^* \in K_\epsilon) + \epsilon. \tag{2.3}$$

From the assumption we see that for any $f \in C_b(M_1(E))$

$$\lim_{n\to\infty}\frac{1}{n}\log\int \mathrm{e}^{nf(\gamma)} P_\mu^{(n)}(Q_n^* \in \mathrm{d}\gamma) = \lim_{n\to\infty}\log\int \mathrm{e}^{f(\gamma)} Q_n^\mu(\mathrm{d}\gamma) = \log\int \mathrm{e}^f\,\mathrm{d}Q_\infty^\mu.$$

Thus from theorem 2.1 in [4] we obtain

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log P_\mu^{(n)}\big(d\big(Q_n^*, Q_\infty^\mu\big) \geqslant \delta;\ Q_n^* \in K_\epsilon\big)\\ \leqslant -\inf\big\{h\big(Q, Q_\infty^\mu\big), d\big(Q, Q_\infty^\mu\big) \geqslant \delta \text{ and } Q \in K_\epsilon\big\}\end{aligned} \tag{2.4}$$

where $h\big(Q, Q_\infty^\mu\big) = \sup_{f\in C_b(M_1(E))}\big[Q(f) - \log Q_\infty^\mu(\mathrm{e}^f)\big]$, which is just the relative entropy of Q w.r.t. Q_∞^μ and assumes 0 only at Q_∞^μ. Thus the rhs of (2.4) is negative. Combining this with (2.3) we conclude

$$\lim_{n\to\infty} P_\mu^{(n)}\big(d\big(Q_n^*, Q_\infty^\mu\big) \geqslant \delta\big) = 0$$

since $\epsilon > 0$ is arbitrary. The theorem is proved. $\square$

The following corollary is a direct consequence of theorems 1 and 2.

Corollary 2.3. *If* $\lim_{t\to\infty} T_t^\mu = \nu$ *weakly for some* $\nu \in M_{\mathrm{i,e}}(E)$, *then* $\lim_{n\to\infty} Q_n^* = \delta_\nu$ *weakly in probability* $P_\mu^{(n)}$. *If* $\mu \in M_i(E)$ *and* Q_∞^μ *is the weak limit of* Q_t^μ *as* $t \to \infty$, *then* $\lim_{n\to\infty} Q_n^* = Q_\infty^\mu$ *weakly in probability* $P_\mu^{(n)}$.

Now we provide conditions for tightness of $\{Q_t^\mu, t \geqslant 0\}$ and $\{P_\mu(Q_n^* \in \cdot), n \geqslant 1\}$.

Proposition 2.4.

(1) If there exists a function ϕ on E that is bounded from below and has compact level sets (i.e., for any real a, $\{\eta, \phi(\eta) \leqslant a\}$ is a compact subset of E), and for some constant $c \geqslant 0$,

$$\int_0^t E^{P_\mu}\phi(\eta_u)\,\mathrm{d}u \leqslant ct \qquad \forall t > 0 \tag{2.5}$$

then $\left\{Q_t^\mu, t \geqslant 0\right\}$ is tight;

(2) If $\left\{Q_n^\mu, n \geqslant 0\right\}$ is tight, then so is $\left\{P_\mu^{(n)}(Q_n^* \in \cdot), n \geqslant 1\right\}$.

Proof.

(1) The proof is standard. Let ϕ be a function satisfying the required conditions, $-b$ be a lower bound of it. For each $l \geqslant 1$, let $E_l = \{\eta \in E, \phi(\eta) + b \leqslant l^3\}$, which is compact in E. Define

$$J_l = \left\{\mu \in M_1(E), \mu\left(E_l^c\right) \leqslant \frac{1}{l}\right\}.$$

Then by the assumption on ϕ,

$$Q_t^\mu\left(J_l^c\right) \leqslant \frac{1}{l^2 t}\left[\int_0^t E^{P_\mu}\phi(\eta_u)\,\mathrm{d}u + bt\right] \leqslant \frac{b+c}{l^2}. \tag{2.6}$$

Thus if we define compact subsets of $M_1(E)$ by

$$K_l = \cap_{i=1}^\infty J_{l+i} \qquad l \geqslant 1$$

then it follows from (2.6) that for every $l \geqslant 1$,

$$Q_t^\mu\left(K_l^c\right) \leqslant \frac{b+c}{l}$$

implying the tightness of $\left\{Q_t^\mu, t \geqslant 0\right\}$.

(2) Now suppose that $\left\{Q_n^\mu, n \geqslant 1\right\}$ is tight. Then it is well known that there is a function Φ on $M_1(E)$, bounded from below with compact level sets, such that for some constant c,

$$Q_n^\mu(\Phi) \leqslant c \qquad \forall n \geqslant 1.$$

This is equivalent to

$$\sum_{i=1}^n E^{P_\mu^{(n)}}\Phi\left(L_n^i\right) \leqslant cn \quad \forall n \geqslant 1$$

which is similar to (2.5). Thus applying the same argument as above we obtain the tightness of $\left\{P_\mu^{(n)}(Q_n^* \in \cdot), n \geqslant 1\right\}$. The proof is completed. □

3. Identifying the components

Now we turn to the problem of identifying the components in a mixture of measures. We will use the empirical fields on E defined by

$$R_n = R_n(\eta) = \frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n}\delta_{\theta_i\eta} \qquad \eta \in E \quad n \geqslant 1$$

where $\{\Lambda_n, n \geqslant 1\}$ is any fixed sequence of finite subsets of Z^d increasing to Z^d as $n \nearrow \infty$, $|\Lambda_n|$ is the cardinality of Λ_n, θ_i is the usual shift operator on E defined by $(\theta_i\eta)(j) = \eta(j+i)$ for $j \in Z^d$. A general result is the following

Theorem 3.1. *Let A_δ, $\delta \geqslant 0$, be a family of closed subsets of $M_1(E)$ with $A_\delta \searrow A_0$ as $\delta \searrow 0$. Given $\mu \in M_1(E)$ satisfying that $\{\mu(R_n \in \cdot), n \geqslant 1\}$ is tight, $E^\mu(R_n)$ converges weakly to μ and that for each compact $K \subset M_1(E)$*

$$\limsup_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\mu\big(R_n \in A_\delta^c \cap K\big) \leqslant \limsup_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\mu(R_n \in A_\delta \cap K). \tag{3.1}$$

Further suppose that there is a sequence $\{a_n, n \geqslant 1\}$ of positive numbers with $a_n \to \infty$, and a function I from $M_1(E)$ to $[0, \infty]$ such that for every compact $K \subset M_1(E)$,

$$\limsup_{n\to\infty}\frac{1}{a_n}\log\mu(R_n \in K) \leqslant -\inf_{\gamma\in K} I(\gamma) \tag{3.2}$$

and that $I(\gamma) = 0$ implies that γ has property (**P**)*. Assume that*

$$\mu = \int_{A_0} \gamma\Gamma(\mathrm{d}\gamma) \tag{3.3}$$

for a unique probability measure Γ on $M_1(E)$ supported in A_0. Then Γ-almost all γ possess property (**P**)*.*

Before proving it, we discuss a potential application of the above theorem. We are interested in the case where μ is translation invariant and is an invariant measure of some interacting stochastic system, or some Markov process, as described in the introduction, and where (3.3) is the ergodic decomposition of μ. In this case A_0 is the support of the mixing measure in the decomposition representation. We choose $A_\delta = \{\gamma, d(\gamma, A_0) \leqslant \delta\}$. Then it is clear

$$\lim_{n\to\infty}\mu(R_n \in A_\delta) = 1$$

for each $\delta > 0$. Thus (3.1) is satisfied. Obviously $\mu = E^\mu(R_n)$. As for tightness, conditions similar to those given in proposition 2.4 are sufficient and practical. Thus what we need to do is to specify the property **(P)** and to find out when (3.2) will be satisfied with some I reflecting **(P)**. A concrete case is as follows:

$d = 1$ and μ is a stationary Markov chain with state space W. Then by the large deviation results for Markov chain (cf, e.g., [5]) we see that under some mild conditions, there is a lower semi-continuous function I from $M_s(E)$ to $[0, \infty]$ such that for every compact subset K of $M_s(E)$,

$$\limsup_{n\to\infty}\frac{1}{n}\mu(R_n \in K) \leqslant -\inf_{\gamma\in K} I(\gamma) \tag{3.4}$$

and that $I(\gamma) = 0$ implies that γ is also a stationary Markov chain with the same transition probabilities as those for μ, the property **(P)** in this case. That is, our argument shows that under the required conditions, every stationary Markov chain on a Polish space is a mixture of those stationary and ergodic Markov chains with the same transition probabilities.

Applications to interacting particle systems will be discussed in section 4.

Proof of theorem 3.1. The argument is similar to that used in proving theorem 2.1. For given $\epsilon > 0$, choose a compact subset K_ϵ of $M_1(E)$, such that

$$\mu\big(R_n \in K_\epsilon^c\big) \leqslant \epsilon \qquad \forall n \geqslant 1. \tag{3.5}$$

Then for any open set $U \supset A_0 \cap \{\gamma, I(\gamma) = 0\}$,

$$\mu(R_n \in U^c) \leqslant \epsilon + \mu(R_n \in U^c \cap K_\epsilon) \qquad \forall n \geqslant 1. \tag{3.6}$$

Note that by assumptions (3.1) and (3.2)

$$\limsup_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{a_n}\log\mu(R_n\in U^c\cap K_\epsilon)\leqslant\limsup_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{a_n}\log\mu(R_n\in A_\delta\cap U^c\cap K_\epsilon)$$
$$\leqslant -\inf_{\gamma\in A_0\cap U^c\cap K_\epsilon} I(\gamma)<0. \tag{3.7}$$

Combining this with (3.5) we obtain

$$\lim_{n\to\infty}\mu(R_n\in U^c)=0$$

since ϵ is arbitrary. This implies that any weak limit Γ_∞ of $\mu(R_n\in\cdot)$ is supported in $A_0\cap\{\gamma, I(\gamma)=0\}$. Let $\{\mu(R_{n_k}\in\cdot), k\geqslant 1\}$ converge weakly to Γ_∞ as $k\to\infty$. Then by the assumptions on μ we see that

$$\mu=\lim_{k\to\infty}E^\mu(R_{n_k})=\lim_{k\to\infty}\int\gamma\mu(R_{n_k}\in \mathrm{d}\gamma)=\int_{A_0\cap\{\gamma,I(\gamma)=0\}}\gamma\Gamma_\infty(\mathrm{d}\gamma)$$

weakly. By the uniqueness assumption on such representation, $\Gamma_\infty=\Gamma$. This proves the theorem. In fact, this also shows that $\lim_{n\to\infty}\mu(R_n\in\cdot)=\Gamma$ weakly. □

4. Applications to particle systems

In this section we sketch some applications of the previous results to interacting particle systems. We mainly consider Gibbs fields and related stochastic Ising models. Since the configuration space involved will be compact, all the required tightness conditions are automatically fulfilled.

We first briefly recall some notions concerning Gibbs measures (for a precise description, see [6, 9]). In this setting W is compact and $E=W^{Z^d}$ is the space of configurations. An interaction potential is a family $\{\Phi_\Lambda : \Lambda\subset Z^d \text{ is finite}\}$ of local functions with Φ_Λ being defined on W^Λ. The potential is assumed to be translation invariant and satisfy

$$\sum_{\Lambda\ni 0}\|\Phi_\Lambda\|<\infty$$

where $\|\ \|$ denotes the sup-norm. For $V\subset Z^d$, $\mathcal{F}_V$ denotes the sigma-algebra generated by the projections $\eta\to\eta(x), x\in V$. A probability measure ν on E is called a Gibbs measure relative to this potential if a version of the conditional distribution of μ on $E_\Lambda=W^\Lambda$ given $\mathcal{F}_{\Lambda^c}$ is given by

$$\mu(\xi_\Lambda|\mathcal{F}_{\Lambda^c})(\eta)=[Z(\eta)]^{-1}\exp\left\{\sum_{A\cap\Lambda\neq\emptyset}\Phi_A(\xi_\Lambda\eta_{\Lambda^c})\right\}$$

where $\xi_\Lambda\eta_{\Lambda^c}$ is the configuration with $(\xi_\Lambda\eta_{\Lambda^c})(x)=\xi(x)$ if $x\in\Lambda$; $=\eta(x)$ if $x\in\Lambda^c$, $Z(\eta)$ is the normalizing constant. Denote by $\mathcal{G}$ the set of Gibbs measures relative to the given potential, $\mathcal{G}_e$ the extremal elements in $\mathcal{G}$. $(\mathcal{G}\cap M_s(E))_e$ is the set of extremal elements in $\mathcal{G}\cap M_s(E)$. The results in the following theorem may be found elsewhere. Here we derive them as corollaries of theorem 3.1.

Theorem 4.1. *Every $\mu\in\mathcal{G}\cap M_s(E)$ admits a unique representation (3.3) in which $A_0=\mathcal{G}\cap M_{s,e}(E)$. In particular*

$$(\mathcal{G}\cap M_s(E))_e=\mathcal{G}\cap M_{s,e}(E).$$

Proof. To fit this situation into the setting of theorem 3.1 and verify the conditions required, we take (3.3) be the standard ergodic decomposition of μ, the property (**P**) be that 'A measure is a Gibbs measure relative to the given potential'. Then, from the discussion following the statement of theorem 3.1, we see that the only thing we need to do is to verify (3.2). But this follows from the well-known large deviation results for Gibbs measures which say that, there is a lower semi-continuous function I from $M_s(E)$ to $[0, \infty]$, such that for every closed $K \subset M_s(E)$, (3.2) holds, and that $I(\gamma) = 0$ iff γ is a Gibbs measure relative to the same potential (cf [3, 6]). Thus A_0 can be represented as $A_0 = \mathcal{G} \cap M_{s,e}(E)$, proving the first conclusion of the theorem. The last conclusion is a direct consequence of this. □

Next, we briefly discuss an application of theorems in section 2 to stochastic Ising models. For detailed definition of such models, see [8]. We only consider the ferromagnetic case here, or more specifically, the nearest-neighbour case for simplicity. In this case, $W = \{-1, 1\}$, and the potential is taken to be $\Phi_\Lambda(\eta) = \beta\eta(x)\eta(y)$ for $\Lambda = \{x, y\}$ with $|y - x| = 1$; $= 0$ for other Λ, where $\beta > 0$ is the parameter representing the inverse temperature. The stochastic Ising model that we are considering here is a spin flip system with spin flip rates given by

$$c(x, \eta) = \exp\left\{-\beta \sum_{|y-x|=1} \eta(x)\eta(y)\right\} \qquad x \in Z^d \quad \eta \in E$$

which represent the probability rates that a configuration changes its state at a single site x. Such a system is formally defined to be a continuous-time Markov process with state space E (cf [8]). Thus we are in the setting discussed in section 2. An important result for such a system is that there are extremal invariant measures μ_- and μ_+ which are translation invariant Gibbs measures relative to the given potential, and there is no phase transition or, equivalently, the system is ergodic in the sense described in paragraph 2 of section 1, iff $\mu_- = \mu_+$. Furthermore, it is also known that

$$\lim_{t\to\infty} \delta_{-1} S(t) = \mu_- \qquad \text{and} \qquad \lim_{t\to\infty} \delta_1 S(t) = \mu_+$$

weakly, where δ_{-1} and δ_1 are the Dirac measures on E centred at the identically -1 and +1 configurations, respectively. It then follows that

$$\lim_{t\to\infty} T_t^{\delta_{-1}} = \mu_- \qquad \text{and} \qquad \lim_{t\to\infty} T_t^{\delta_1} = \mu_+$$

weakly. Thus from corollary 2.3 we know that

$$\lim_{t\to\infty} Q_n^* = \delta_{\mu_-} \qquad (\text{resp. } \delta_{\mu_+})$$

in probability P_{-1} (resp. in P_1). These imply that for every $f \in C_b(E)$,

$$\lim_{t\to\infty} \frac{1}{n^2} \sum_{i=1}^n \int_0^n f\left(\eta_t^{(i)}\right) \mathrm{d}t = \mu_-(f) \qquad (\text{resp. } \mu_+(f))$$

in probability P_{-1} (resp. in P_1). In particular, we have

$$\lim_{t\to\infty} \frac{1}{n^2} \sum_{i=1}^n \int_0^n \eta_t^{(i)}(0)\, \mathrm{d}t = \mu_-(\eta(0) = 1) \qquad (\text{resp. } \mu_+(\eta(0) = 1))$$

in probability P_{-1} (resp. in P_1). These suggest a way of estimating the two extremal measures $\mu_\pm$ and especially, of inferring the existence of phase transition.

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